

# The FRT-Construction via Quantum Affine Algebras and Smash Products

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**ABSTRACT.** For every element  $w$  in the Weyl group of a simple Lie algebra  $\mathfrak{g}$ , De Concini, Kac, and Procesi defined a subalgebra  $\mathcal{U}_q^w$  of the quantized universal enveloping algebra  $\mathcal{U}_q(\mathfrak{g})$ . The algebra  $\mathcal{U}_q^w$  is a deformation of the universal enveloping algebra  $\mathcal{U}(\mathfrak{n}_+ \cap w \cdot \mathfrak{n}_-)$ . We construct smash products of certain finite-type De Concini-Kac-Procesi algebras to obtain ones of affine type; we have analogous constructions in types  $A_n$  and  $D_n$ . We show that the multiplication in the affine type De Concini-Kac-Procesi algebras arising from this smash product construction can be twisted by a cocycle to produce certain subalgebras related to the corresponding Faddeev-Reshetikhin-Takhtajan bialgebras.

## 1. Introduction

Let  $k$  be an infinite field and suppose an algebraic  $k$ -torus  $H$  acts rationally on a noetherian  $k$ -algebra  $A$  by  $k$ -algebra automorphisms. Goodearl and Letzter [9] showed that  $\text{spec}(A)$  is partitioned into strata indexed by the  $H$ -invariant prime ideals of  $A$ . Furthermore, they showed that each stratum is homeomorphic to the prime spectrum of a Laurent polynomial ring. The Goodearl-Letzter stratification results apply to the case when  $A$  is an iterated Ore extension under some assumptions relating the action of  $H$  to the structure of  $A$ . In this setting Cauchon's deleting derivations algorithm [4] gives an iterative procedure for classifying the  $H$ -primes. After several such algebras were studied, such as the algebras of quantum matrices  $\mathcal{O}_q(M_{\ell,p}(k))$  [4, 8, 13], it was noticed that many of these algebras fall into the setting of De Concini-Kac-Procesi algebras [6].

The De Concini-Kac-Procesi algebras are subalgebras of quantized universal enveloping algebras  $\mathcal{U}_q(\mathfrak{g})$  associated to the elements of the corresponding Weyl group  $W_{\mathfrak{g}}$ . They may be viewed as deformations of the universal enveloping algebra  $\mathcal{U}(\mathfrak{n}_+ \cap w \cdot \mathfrak{n}_-)$ , where  $\mathfrak{n}_+$  and  $\mathfrak{n}_-$  are the positive and negative nilpotent Lie subalgebras of  $\mathfrak{g}$ , respectively. Mériaux and Cauchon [15] and Yakimov [16] recently proved that the poset of  $H$ -primes of a De Concini-Kac-Procesi algebra  $\mathcal{U}_q^w$  ordered under inclusion is isomorphic to the poset  $W^{\leq w}$  of Weyl group elements less than or equal to  $w$  under the Bruhat ordering. In [16], Yakimov also gives explicit generating sets for the  $H$ -primes in terms of Demazure modules.

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In this paper we introduce three type- $D$  algebras. The first algebra is obtained from a De Concini-Kac-Procesi algebra of type  $\mathfrak{so}_{2n+2}$  and a smash product construction. We show that this algebra is isomorphic to our second algebra, a De Concini-Kac-Procesi algebra associated to the affine Weyl group of type  $\widehat{D}_{n+1}$ . Finally, we show that twisting the multiplication in these algebras by a certain 2-cocycle produces algebras related to the type- $D$  Faddeev-Reshetikhin-Takhtajan bialgebras. In the last section, we produce analogous results with type- $A$  algebras. In a forthcoming publication we will return to the  $H$ -spectrum of these algebras.

Section 3 introduces the first of these algebras, an algebra which resembles a smash product of a De Concini-Kac-Procesi algebra with itself. Let  $k$  be an algebraically closed field of characteristic zero and suppose  $q \in k$  is not a root of unity. Fix an integer  $n \geq 3$ . Let  $W(D_{n+1})$  be the Weyl group of type  $D_{n+1}$  with standard generating set  $\{s_1, \dots, s_{n+1}\}$  and let

$$(1.1) \quad w_n = (s_{n+1}s_n \cdots s_2s_1)(s_3s_4 \cdots s_ns_{n+1}) \in W(D_{n+1}).$$

Let  $\mathcal{U}_D^{\geq 0}$  denote the quantized positive Borel algebra of type  $D_{n+1}$  and let  $\mathcal{U}_q^{w_n}$  be the De Concini-Kac-Procesi subalgebra of  $\mathcal{U}_D^{\geq 0}$  corresponding to  $w_n$ . In fact,  $\mathcal{U}_q^{w_n}$  is isomorphic to  $\mathcal{O}_q(\mathfrak{ok}^{2n})$ , the algebra of even-dimensional quantum Euclidean space. We define an action  $\lambda$  of  $\mathcal{U}_D^{\geq 0}$  on  $\mathcal{U}_q^{w_n}$ , which is a modification of the adjoint action of the Hopf algebra  $\mathcal{U}_D^{\geq 0}$  on itself. This action equips  $\mathcal{U}_q^{w_n}$  with the structure of a left  $\mathcal{U}_D^{\geq 0}$ -module algebra. We then consider the smash product  $\mathcal{U}_q^{w_n} \# \mathcal{U}_D^{\geq 0}$  with respect to  $\lambda$  and set  $(\mathcal{U}_q^{w_n})^\#$  to be the subalgebra of  $\mathcal{U}_q^{w_n} \# \mathcal{U}_D^{\geq 0}$  generated by  $\{u\#1, 1\#u \mid u \in \mathcal{U}_q^{w_n}\}$ . This is the first of the three type- $D$  algebras.

In Section 4 we introduce a second type- $D$  algebra; it is a De Concini-Kac-Procesi algebra of affine type. Let  $W(\widehat{D}_{n+1})$  denote the affine Weyl group of type  $\widehat{D}_{n+1}$  with generating set  $\{s_0, s_1, \dots, s_{n+1}\}$  and let

$$(1.2) \quad \widehat{w}_n = (s_{n+1} \cdots s_1)(s_3 \cdots s_{n+1})s_0(s_n \cdots s_3)(s_1 \cdots s_n)s_0 \in W(\widehat{D}_{n+1}).$$

The main result of this section is Theorem 4.4, where we prove that the algebras  $\mathcal{U}_q^{\widehat{w}_n}$  and  $(\mathcal{U}_q^{w_n})^\#$  are isomorphic.

In Section 5 we introduce an algebra which we denote by  $\mathbb{X}_{n,q}$ . We show that  $\mathbb{X}_{n,q}$  is related to the bialgebra  $\mathcal{A}(R_{D_n})$  arising from the type- $D_n$  FRT construction. In particular, we label the standard generators of  $\mathcal{A}(R_{D_n})$  by  $Y_{ij}$ , for  $1 \leq i, j \leq 2n$ , and let  $T_{2,n} \subseteq \mathcal{A}(R_{D_n})$  be the subalgebra generated by  $\{Y_{ij} : 1 \leq i \leq 2, 1 \leq j \leq 2n\}$  and observe that there is a surjective algebra homomorphism  $\mathbb{X}_{n,q} \rightarrow T_{2,n}$  (see Proposition 5.1). We thus refer to  $\mathbb{X}_{n,q}$  as a *parent* of  $T_{2,n}$ . Finally, in Thm. 5.2 we prove that  $\mathbb{X}_{n,q}$  is isomorphic to a cocycle twist (in the sense of [1]) of  $\mathcal{U}_q^{\widehat{w}_n}$ . From this, it follows that  $\mathbb{X}_{n,q}$  is an iterated Ore extension over  $k$ .

In Section 6 we proceed to demonstrate analogous results in the type  $A_m$  setting. We fix an integer  $m > 1$  and let  $W(A_m)$  be the Weyl group of type  $A_m$  with generating set  $\{s_1, \dots, s_m\}$ . Let

$$(1.3) \quad c_m = s_1 \cdots s_m \in W(A_m)$$

denote a Coxeter element. Notice that the De Concini-Kac-Procesi algebra  $\mathcal{U}_q^{c_m}$  is isomorphic to  $\mathcal{O}_q(k^m)$ , the algebra known as quantum affine space. Quantum euclidean space, seen in Section 3, can be thought of as a type- $D$  analogue of quantum affine space. Let  $\mathcal{U}_A^{\geq 0}$  denote the quantized positive Borel algebra of type  $A_m$ . We define an action  $\lambda_A : \mathcal{U}_A^{\geq 0} \otimes \mathcal{U}_q^{c_m} \rightarrow \mathcal{U}_q^{c_m}$  endowing  $\mathcal{U}_q^{c_m}$  with the structure

of a left  $\mathcal{U}_A^{\geq 0}$ -module algebra and define  $(\mathcal{U}_q^{c_m})^\#$  to be the subalgebra of  $\mathcal{U}_q^{c_m} \# \mathcal{U}_A^{\geq 0}$  generated by  $\{1 \# u, u \# 1 \mid u \in \mathcal{U}_q^{c_m}\}$ . Finally, we let  $W(\hat{A}_m)$  denote the affine Weyl group of type  $\hat{A}_m$  with generating set  $\{s_0, s_1, \dots, s_m\}$  and let

$$(1.4) \quad \hat{c}_m = (s_1 \cdots s_m)(s_0 s_1 \cdots s_{m-1}) \in W(\hat{A}_m).$$

In Thm. 6.4, we prove that the corresponding De Concini-Kac-Procesi algebra  $\mathcal{U}_q^{\hat{c}_m}$  is isomorphic to  $(\mathcal{U}_q^{c_m})^\#$ . We further demonstrate that  $\mathcal{U}_q^{\hat{c}_m}$  is isomorphic to a cocycle twist of  $\mathcal{O}_q(M_{2,m})$ , the algebra of  $2 \times m$  quantum matrices. We see  $\mathbb{X}_{m,q}$  as a “type D” analogue of  $\mathcal{O}_q(M_{2,m})$  because  $\mathcal{O}_q(M_{2,m})$  is a subalgebra of the FRT-bialgebra  $\mathcal{A}(R_{A_{m-1}}) \cong \mathcal{O}_q(M_m(k))$ . The key distinction is that  $\mathcal{O}_q(M_{2,m}(k))$  is a subalgebra of  $\mathcal{A}(R_{A_{m-1}})$ , whereas  $\mathbb{X}_{m,q}$  is a parent of the analogous subalgebra  $T_{2,m} \subseteq \mathcal{A}(R_{D_m})$ .

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## 2. Preliminaries

Let  $Q$  be a  $\mathbb{Z}$ -module with basis  $\Pi = \{\alpha_1, \dots, \alpha_n\}$ . Suppose  $\langle \cdot, \cdot \rangle$  is a symmetric form on  $Q$  with

$$(2.1) \quad c_{ij} := 2 \langle \alpha_i, \alpha_j \rangle / \langle \alpha_i, \alpha_i \rangle$$

non-positive integers for every  $i, j \in \{1, \dots, n\}$  with  $i \neq j$ , and there exist coprime positive integers  $d_1, \dots, d_n$  so that the matrix  $(d_i c_{ij})$  is symmetric. Let  $k$  be a field and let  $q \in k$  be nonzero. Set  $q_i = q^{d_i}$  and assume  $q_i \neq \pm 1$ . To the triple  $(Q, \Pi, q)$ , we have an associated quantized enveloping algebra  $\mathcal{U}$ . As a  $k$ -algebra,  $\mathcal{U}$  is generated by  $u_1^\pm, \dots, u_n^\pm$  and  $\{v_\mu : \mu \in Q\}$  and has defining relations

$$(2.2) \quad v_0 = 1, \quad v_\mu v_\rho = v_{\mu+\rho}, \quad (\mu, \rho \in Q),$$

$$(2.3) \quad v_\mu u_i^\pm = q^{\pm \langle \mu, \alpha_i \rangle} u_i^\pm v_\mu, \quad (\mu \in Q, i \in \{1, \dots, n\}),$$

$$(2.4) \quad u_i^+ u_j^- = u_j^- u_i^+ + \delta_{ij} \frac{v_{\alpha_i} - v_{-\alpha_i}}{q_i - q_i^{-1}}, \quad (i, j \in \{1, \dots, n\}),$$

$$(2.5) \quad \sum_{r=0}^{1-c_{ij}} (-1)^r \begin{bmatrix} 1 - c_{ij} \\ r \end{bmatrix}_{q_i} (u_i^\pm)^{1-c_{ij}-r} u_j^\pm (u_i^\pm)^r = 0, \quad (i \neq j).$$

Here,

$$(2.6) \quad [\ell]_{q_i} = \frac{q_i^\ell - q_i^{-\ell}}{q_i - q_i^{-1}}, \quad [\ell]_{q_i}! = [1]_{q_i} \cdots [\ell]_{q_i}, \quad \begin{bmatrix} \ell \\ m \end{bmatrix}_{q_i} = \frac{[\ell]_{q_i}!}{[m]_{q_i}! [\ell - m]_{q_i}!}.$$

Furthermore,  $\mathcal{U}$  has a Hopf algebra structure with comultiplication  $\Delta$ , antipode  $S$ , and counit  $\epsilon$  maps given by

$$(2.7) \quad \Delta(u_i^+) = v_{-\alpha_i} \otimes u_i^+ + u_i^+ \otimes 1, \quad \Delta(v_\mu) = v_\mu \otimes v_\mu, \quad \Delta(u_i^-) = 1 \otimes u_i^- + u_i^- \otimes v_{\alpha_i},$$

$$(2.8) \quad S(u_i^+) = -v_{\alpha_i} E_i, \quad S(v_\mu) = v_{-\mu}, \quad S(u_i^-) = -u_i^- v_{-\alpha_i},$$

$$(2.9) \quad \epsilon(u_i^+) = 0, \quad \epsilon(v_\mu) = 1, \quad \epsilon(u_i^-) = 0,$$

for every  $\mu \in Q$  and  $1 \leq i \leq n$ .

For every  $i \in \{1, \dots, n\}$ , we let  $s_i : Q \rightarrow Q$  be the simple reflection

$$(2.10) \quad s_i : \mu \rightarrow \mu - \frac{2 \langle \mu, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i,$$

and let  $W = \langle s_1, \dots, s_n \rangle$  denote the Weyl group. The standard presentation for the braid group  $B$  is the generating set  $\{T_w : w \in W\}$  subject to the relations  $T_w T_{w'} = T_{ww'}$  for every  $w, w' \in W$  satisfying  $\ell(w) + \ell(w') = \ell(ww')$ , where  $\ell$  is the length function on  $W$ . For each  $i \in \{1, \dots, n\}$ , we set  $T_i := T_{s_i}$ . Thus, the braid group  $B$  is generated by  $T_1, \dots, T_n$ . When  $q$  is not a root of unity,  $B$  acts via algebra automorphisms on  $\mathcal{U}$  as follows:

$$(2.11) \quad T_i v_\mu = v_{s_i(\mu)}, \quad T_i u_i^+ = -u_i^- v_{\alpha_i}, \quad T_i u_i^- = -v_{-\alpha_i} u_i^+,$$

$$(2.12) \quad T_i u_j^+ = \sum_{r=0}^{-c_{ij}} \frac{(-q_i)^{-r}}{[-c_{ij}-r]_{q_i}! [r]_{q_i}!} (u_i^+)^{-c_{ij}-r} u_j^+ (u_i^+)^r, \quad (i \neq j),$$

$$(2.13) \quad T_i u_j^- = \sum_{r=0}^{-c_{ij}} \frac{(-q_i)^r}{[-c_{ij}-r]_{q_i}! [r]_{q_i}!} (u_i^-)^r u_j^- (u_i^-)^{-c_{ij}-r}, \quad (i \neq j).$$

for all  $i, j \in \{1, \dots, n\}$ ,  $\mu \in Q$  [14].

Fix  $w \in W$ . For a reduced expression

$$(2.14) \quad w = s_{i_1} \cdots s_{i_t}$$

define the roots

$$(2.15) \quad \beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1} \alpha_{i_2}, \dots, \beta_t = s_{i_1} \cdots s_{i_{t-1}} \alpha_{i_t}$$

and the root vectors

$$(2.16) \quad X_{\beta_1} = u_{i_1}^+, X_{\beta_2} = T_{s_{i_1}} u_{i_2}^+, \dots, X_{\beta_t} = T_{s_{i_1}} \cdots T_{s_{i_{t-1}}} u_{i_t}^+.$$

Following [6], let  $\mathcal{U}_q^w$  denote the subalgebra of  $\mathcal{U}$  generated by the root vectors  $X_{\beta_1}, \dots, X_{\beta_t}$  (depends on the reduced expression).

When  $k$  is algebraically closed of characteristic zero and  $q$  is not a root of unity, De Concini, Kac, and Procesi proved the following:

**Theorem 2.1.** [6, Proposition 2.2] *If  $(c_{ij})$  is a finite-type Cartan matrix, then the algebra  $\mathcal{U}_q^w$  does not depend on the reduced expression for  $w$ . The algebra  $\mathcal{U}_q^w$  has the PBW basis*

$$X_{\beta_1}^{n_1} \cdots X_{\beta_t}^{n_t}, \quad n_1, \dots, n_t \in \mathbb{Z}_{\geq 0}.$$

Beck later proved the analogous result for the case when  $(c_{ij})$  is an affine Cartan matrix (with  $q$  transcendental over  $\mathbb{Q}$ ) [3].

### 3. A smash product of type $D_{n+1}$

**3.1. The Algebras  $\mathcal{U}_q(\mathfrak{so}_{2n+2})$ ,  $\mathcal{U}_q^{w_n}$ , and  $\mathcal{O}_q(\mathfrak{ok}^{2n})$ .** Fix an integer  $n \geq 3$ , and let  $Q(D_{n+1})$  be the additive abelian subgroup of  $\mathbb{R}^{n+1}$  consisting of the vectors having integer-valued coordinates  $(a_1, \dots, a_{n+1})$  with the sum  $\sum a_i$  being an even number. Let  $\langle \cdot, \cdot \rangle$  denote the restriction of the standard inner product on  $\mathbb{R}^{n+1}$  (i.e.  $\langle e_i, e_j \rangle = \delta_{ij}$ ) to  $Q(D_{n+1})$ . The group  $Q(D_{n+1})$  is generated by the positive simple roots  $\alpha_i = e_i - e_{i-1}$  for  $2 \leq i \leq n+1$  and  $\alpha_1 = e_1 + e_2$ . For a positive simple root  $\alpha_i$ , let  $s_i$  denote the corresponding simple reflection and let  $W(D_{n+1}) = \langle s_1, \dots, s_{n+1} \rangle$  denote the associated Weyl group. The associated Cartan matrix  $(c_{ij})$  is symmetric.

Hence  $d_1 = \dots = d_{n+1} = 1$ . Therefore, the parameters  $q_1, \dots, q_{n+1}$  are all equal to  $q$ . As usual, we put  $\hat{q} = q - q^{-1}$ . Let  $\mathcal{U}_q(\mathfrak{so}_{2n+2})$  denote the corresponding quantized universal enveloping algebra. We label the generators of  $\mathcal{U}_q(\mathfrak{so}_{2n+2})$  by  $E_1, \dots, E_{n+1}, F_1, \dots, F_{n+1}$  and  $\{K_\mu : \mu \in Q(D_{n+1})\}$  and the defining relations are

$$(3.1) \quad K_0 = 1, \quad K_\mu K_\lambda = K_{\mu+\lambda},$$

$$(3.2) \quad K_\mu E_i = q^{\langle \mu, \alpha_i \rangle} E_i K_\mu, \quad K_\mu F_i = q^{-\langle \mu, \alpha_i \rangle} F_i K_\mu,$$

$$(3.3) \quad E_i E_j = E_j E_i, \quad F_i F_j = F_j F_i, \quad (\langle \alpha_i, \alpha_j \rangle = 0 \text{ or } 2),$$

$$(3.4) \quad E_i [E_i, E_j] = q [E_i, E_j] E_i, \quad F_i [F_i, F_j] = q [F_i, F_j] F_i, \quad (\langle \alpha_i, \alpha_j \rangle = -1),$$

$$(3.5) \quad E_i F_j = F_j E_i + \frac{\delta_{ij}}{\hat{q}} (K_{\alpha_i} - K_{-\alpha_i}),$$

for every  $i, j \in \{1, \dots, n+1\}$  and  $\mu, \lambda \in Q(D_{n+1})$ . Here we use the  $q^{-1}$ -commutators, defined by

$$[u, v] := uv - q^{-1}vu$$

for every  $u, v \in \mathcal{U}_q(\mathfrak{so}_{2n+2})$ .

Let  $w_0$  denote the longest element of  $W(D_{n+1})$  and let  $w_0^L$  be the longest element of the parabolic subgroup  $\langle s_1, \dots, s_n \rangle \subseteq W(D_{n+1})$ . Put  $w_n = w_0^L w_0$ . We have a reduced expression

$$(3.6) \quad w_n = (s_{n+1} \cdots s_2 s_1)(s_3 \cdots s_n s_{n+1}) \in W(D_{n+1})$$

and root vectors

$$(3.7) \quad X_{e_{n+1}-e_n}, X_{e_{n+1}-e_{n-1}}, \dots, X_{e_{n+1}-e_1}, X_{e_{n+1}+e_1}, X_{e_{n+1}+e_2}, \dots, X_{e_{n+1}+e_n}.$$

For brevity, we put  $x_i = X_{e_{n+1}-e_i}$  and  $y_i = X_{e_{n+1}+e_i}$  for every  $i \in \{1, \dots, n\}$ . Let  $\mathcal{U}_q^{w_n}$  denote the corresponding DeConcini-Kac-Procesi algebra.

The following can be found in [10], Section 5.6.a.

**Theorem 3.1.** *The algebra  $\mathcal{U}_q^{w_n}$  is isomorphic to the even-dimensional quantum Euclidean space  $\mathcal{O}_q(\mathfrak{ok}^{2n})$ .*

PROOF. We observe that the root vectors  $x_1, \dots, x_n, y_1, \dots, y_n$  of  $\mathcal{U}_q^{w_n}$  can be written inductively as  $x_n = E_{n+1}, y_1 = [x_2, E_1]$  and

$$(3.8) \quad x_i = [x_{i+1}, E_{i+1}],$$

$$(3.9) \quad y_{i+1} = [y_i, E_{i+1}],$$

for all  $1 \leq i < n$ . Using these identities, one can readily check that the root vectors satisfy the defining relations of  $\mathcal{O}_q(\mathfrak{ok}^{2n})$  (c.f. [12, Section 9.3.2]),

$$(3.10) \quad x_i x_j = q^{-1} x_j x_i, \quad y_i y_j = q y_j y_i, \quad (1 \leq i < j \leq n),$$

$$(3.11) \quad x_i y_j = q^{1-\delta_{ij}} y_j x_i + \delta_{ij} \hat{q} \sum_{r=1}^{i-1} (-q)^{i-r-1} x_r y_r, \quad (i, j \in \{1, \dots, n\}).$$

Since  $\mathcal{U}_q^{w_n}$  has a PBW basis of ordered monomials, Eqns. 3.10 and 3.11 are the defining relations. Hence,  $\mathcal{U}_q^{w_n} \cong \mathcal{O}_q(\mathfrak{ok}^{2n})$ .  $\square$

**3.2.  $\mathcal{U}_q^{w_n}$  as a left  $\mathcal{U}_D^{\geq 0}$ -module algebra.** Let  $\mathcal{U}_D^{\geq 0}$  be the sub-Hopf algebra of  $\mathcal{U}_q(\mathfrak{so}_{2n+2})$  generated by  $E_1, \dots, E_{n+1}$ , and  $K_\mu$  for all  $\mu \in Q(D_{n+1})$ . We let  $\pi : \mathcal{U}_D^{\geq 0} \rightarrow \mathcal{U}_D^{\geq 0}$  be the unique algebra map such that

$$(3.12) \quad \pi(E_{n+1}) = 0,$$

$$(3.13) \quad \pi(E_i) = E_i \quad (i \leq n),$$

$$(3.14) \quad \pi(K_\mu) = K_\mu \quad (\mu \in Q(D_{n+1})).$$

We define a function  $\lambda : \mathcal{U}_D^{\geq 0} \otimes \mathcal{U}_q^{w_n} \rightarrow \mathcal{U}_D^{\geq 0}$  by the following sequence of linear maps:

$$(3.15) \quad \lambda : \mathcal{U}_D^{\geq 0} \otimes \mathcal{U}_q^{w_n} \xrightarrow{\text{incl.}} \left(\mathcal{U}_D^{\geq 0}\right)^{\otimes 2} \xrightarrow{\pi \otimes \text{id}} \left(\mathcal{U}_D^{\geq 0}\right)^{\otimes 2} \xrightarrow{\text{adjoint}} \mathcal{U}_D^{\geq 0}$$

and have the following:

**Theorem 3.2.** *For the function  $\lambda$  above, we have  $\text{Im}(\lambda) \subseteq \mathcal{U}_q^{w_n}$ . In particular,  $\lambda$  endows  $\mathcal{U}_q^{w_n}$  with the structure of a left  $\mathcal{U}_D^{\geq 0}$ -module algebra.*

PROOF. For brevity, we set  $u.v = \lambda(u \otimes v)$  for every  $u \in \mathcal{U}_D^{\geq 0}$  and  $v \in \mathcal{U}_q^{w_n}$ . One can verify that

$$(3.16) \quad E_j.x_r = \begin{cases} -q(\delta_{1r}y_2 + \delta_{2r}y_1), & (j = 1), \\ -q\delta_{jr}x_{r-1}, & (j \neq 1), \end{cases}$$

$$(3.17) \quad E_j.y_r = \begin{cases} 0, & (j = n+1), \\ -q\delta_{j,r+1}y_{r+1}, & (j \neq n+1), \end{cases}$$

for all  $r \in \{1, \dots, n\}$ ,  $j \in \{1, \dots, n+1\}$ . Since  $\mathcal{U}_D^{\geq 0}$  is a left  $\mathcal{U}_D^{\geq 0}$ -module algebra (with respect to the adjoint action), the equations 3.16 and 3.17 above, together with the fact that the  $K_\mu$ 's act diagonally on  $\mathcal{U}_q^{w_n}$ , prove the desired result.  $\square$

Using the action map  $\lambda$ , we form the smash product algebra  $\mathcal{U}_q^{w_n} \# \mathcal{U}_D^{\geq 0}$  and define the following subalgebra

$$(3.18) \quad (\mathcal{U}_q^{w_n})^\# := \langle 1\#u, u\#1 \mid u \in \mathcal{U}_q^{w_n} \rangle \subseteq \mathcal{U}_q^{w_n} \# \mathcal{U}_D^{\geq 0}.$$

Loosely speaking, we can think of  $(\mathcal{U}_q^{w_n})^\#$  as being a smash product of  $\mathcal{U}_q^{w_n}$  with itself. Observe for example that  $(\mathcal{U}_q^{w_n})^\#$  is isomorphic as a vector space to  $\mathcal{U}_q^{w_n} \otimes \mathcal{U}_q^{w_n}$ .

**3.3. A Presentation of  $(\mathcal{U}_q^{w_n})^\#$ .** We will spend the rest of this section giving an explicit presentation for the algebra  $(\mathcal{U}_q^{w_n})^\#$  because this will be necessary for proving the main result of Section 4 (Thm. 4.4).

The algebra  $(\mathcal{U}_q^{w_n})^\#$  is generated by  $1\#x_i$ ,  $1\#y_i$ ,  $x_i\#1$ ,  $y_i\#1$  for  $i \in \{1, \dots, n\}$ . To compute the relations among these generators, we need comultiplication formulas for the root vectors  $x_1, \dots, x_n, y_1, \dots, y_n \in \mathcal{U}_q^{w_n}$ . First, we must introduce the elements  $\epsilon_{ij}, E_{r\downarrow s}, E_{s\uparrow r} \in \mathcal{U}_D^{\geq 0}$  for every  $i, j \in \{1, \dots, n\}$  and  $r, s \in \{1, \dots, n+1\}$  with

$r \geq s$ . They are defined recursively via

$$(3.19) \quad E_{r \downarrow s} = \begin{cases} E_r, & (r = s), \\ [E_{r \downarrow s+1}, E_s], & (r \neq s), \end{cases}$$

$$(3.20) \quad E_{s \uparrow r} = \begin{cases} E_s, & (r = s), \\ [E_{s \uparrow r-1}, E_r], & (r \neq s), \end{cases}$$

$$(3.21) \quad \epsilon_{1j} = \begin{cases} 0, & (j = 1), \\ T_j T_{j-1} \cdots T_2 E_1, & (j \neq 1), \end{cases}$$

$$(3.22) \quad \epsilon_{i+1,j} = \begin{cases} [\epsilon_{ij}, E_{i+1}], & (j \neq i, i+1), \\ q\epsilon_{i,i+1}E_{i+1} - q^{-1}E_{i+1}\epsilon_{i,i+1}, & (j = i+1), \\ \epsilon_{i,i+1} + q^{-1}(\epsilon_{ii}E_{i+1} - E_{i+1}\epsilon_{ii}), & (j = i). \end{cases}$$

We have the following:

**Lemma 3.3.** *For every  $i \in \{1, \dots, n\}$ ,*

$$(3.23) \quad \Delta(x_i) = K_{-\deg(x_i)} \otimes x_i + x_i \otimes 1 + \hat{q} \sum_{j=i+1}^n E_{j \downarrow i+1} K_{-\deg(x_j)} \otimes x_j,$$

$$(3.24) \quad \Delta(y_i) = K_{-\deg(y_i)} \otimes y_i + y_i \otimes 1 + \hat{q} \left( \sum_{j=1}^n \epsilon_{ij} K_{-\deg(x_j)} \otimes x_j + \sum_{j=1}^{i-1} E_{j+1 \uparrow i} K_{-\deg(y_j)} \otimes y_j \right).$$

PROOF. Use the induction formulas from Eqns. 3.8 and 3.9 together with the comultiplication formula given in Equation 2.7.  $\square$

From Eqns. 3.16 and 3.17 it follows that for all  $i, j, r \in \{1, \dots, n\}$ ,

$$(3.25) \quad E_{j \downarrow i+1} \cdot x_r = -q\delta_{jr}x_i, \quad E_{j \downarrow i+1} \cdot y_r = (-q)^{j-i}\delta_{ir}y_j,$$

$$(3.26) \quad E_{j+1 \uparrow i} \cdot x_r = (-q)^{i-j}\delta_{ir}x_j, \quad E_{j+1 \uparrow i} \cdot y_r = -q\delta_{jr}y_i,$$

$$(3.27) \quad \epsilon_{ij} \cdot x_r = (-q)^{i+j-2}q^{\delta_{ij}}\delta_{ir}y_j - q\delta_{jr}y_i, \quad \epsilon_{ij} \cdot y_r = 0.$$

Using the identities 3.25-3.27 together with the comultiplication formulas, 3.23-3.24, we compute the following “cross-relations” in  $(\mathcal{U}_q^{w_n})^\#$ .

**Proposition 3.4.** *For every  $i, j \in \{1, \dots, n\}$ ,*

$$(3.28) \quad (1 \# x_i)(x_j \# 1) = \begin{cases} q^{-1}x_j \# x_i - q^{-1}\hat{q}x_i \# x_j, & i < j, \\ q^{-2}x_j \# x_i, & i = j, \\ q^{-1}x_j \# x_i, & i > j, \end{cases}$$

$$(3.29) \quad (1 \# y_i)(y_j \# 1) = \begin{cases} q^{-1}y_j \# y_i - q^{-1}\hat{q}y_i \# y_j, & i > j, \\ q^{-2}y_j \# y_i, & i = j, \\ q^{-1}y_j \# y_i, & i < j, \end{cases}$$

$$(3.30) \quad (1 \# y_i)(x_j \# 1) = q^{-1+\delta_{ij}}x_j \# y_i - \hat{q}q^{-1}y_i \# x_j \\ + \hat{q}q^{-1}\delta_{ij} \left( \sum_{m=1}^n (-q)^{i+m-2} y_m \# x_m + \sum_{m=1}^{i-1} (-q)^{i-m} x_m \# y_m \right),$$

$$(3.31) \quad (1 \# x_i)(y_j \# 1) = q^{-1+\delta_{ij}}y_j \# x_i + \hat{q}q^{-1}\delta_{ij} \sum_{m=i+1}^n (-q)^{m-i} y_m \# x_m.$$

We have the following presentation for  $(\mathcal{U}_q^{w_n})^\#$ :

**Theorem 3.5.** *The algebra  $(\mathcal{U}_q^{w_n})^\#$  is generated by  $1 \# x_i, x_i \# 1$  for  $1 \leq i \leq n$ , and its defining relations are Eqns. 3.28-3.31 together with the relations*

$$(3.32) \quad (1 \# x_i)(1 \# x_j) = q^{-1}(1 \# x_j)(1 \# x_i), \quad (1 \leq i < j \leq n),$$

$$(3.33) \quad (1 \# y_i)(1 \# y_j) = q(1 \# y_j)(1 \# y_i), \quad (1 \leq i < j \leq n),$$

$$(3.34) \quad (1 \# x_i)(1 \# y_j) = q^{1-\delta_{ij}}(1 \# y_j)(1 \# x_i) \\ + \delta_{ij}\hat{q} \sum_{r=1}^{i-1} (-q)^{i-r-1} (1 \# x_r)(1 \# y_r), \quad (i, j \in \{1, \dots, n\}),$$

$$(3.35) \quad (x_i \# 1)(x_j \# 1) = q^{-1}(x_j \# 1)(x_i \# 1), \quad (1 \leq i < j \leq n),$$

$$(3.36) \quad (y_i \# 1)(y_j \# 1) = q(y_j \# 1)(y_i \# 1), \quad (1 \leq i < j \leq n),$$

$$(3.37) \quad (x_i \# 1)(y_j \# 1) = q^{1-\delta_{ij}}(y_j \# 1)(x_i \# 1) \\ + \delta_{ij}\hat{q} \sum_{r=1}^{i-1} (-q)^{i-r-1} (x_r \# 1)(y_r \# 1), \quad (i, j \in \{1, \dots, n\}).$$

PROOF. The generators  $1 \# x_1, \dots, 1 \# x_n$  generate a subalgebra isomorphic to  $\mathcal{U}_q^{w_n}$ , as do the generators  $x_1 \# 1, \dots, x_n \# 1$ , giving us the relations 3.32-3.37. The universal property of smash products (for example, see [11, Section 1.8]) and the PBW basis of De Concini-Kac-Procesi algebras imply that the cross relations of 3.28-3.31 together with the above relations are a presentation of  $(\mathcal{U}_q^{w_n})^\#$ .  $\square$

#### 4. The quantum affine algebra $\mathcal{U}_q^{\hat{w}_n}$

Let  $Q(\hat{D}_{n+1}) = Q(D_{n+1}) \oplus \mathbb{Z}$  denote the root lattice of type  $\hat{D}_{n+1}$ . As an abelian group,  $Q(\hat{D}_{n+1})$  is generated additively by the positive simple roots  $\alpha_0 := -e_{n+1} - e_n + 1$ ,  $\alpha_1 := e_1 + e_2$ , and  $\alpha_i := e_i - e_{i-1}$  for  $2 \leq i \leq n+1$ . We extend the bilinear form  $\langle \cdot, \cdot \rangle$  on  $Q(D_{n+1})$  to  $Q(\hat{D}_{n+1})$  by setting  $1 \in Q(\hat{D}_{n+1})$  to be isotropic. As before, let  $s_i$  denote the corresponding simple reflection  $s_i : Q(\hat{D}_{n+1}) \rightarrow Q(\hat{D}_{n+1})$ , for  $0 \leq i \leq n+1$ , and  $W(\hat{D}_{n+1}) = \langle s_0, \dots, s_{n+1} \rangle$  is the Weyl



group. The corresponding quantized enveloping algebra  $\mathcal{U}_q(\widehat{\mathfrak{so}}_{2n+2})$  is generated by  $E_0, \dots, E_{n+1}, F_0, \dots, F_{n+1}$  and  $\{K_\mu : \mu \in Q(\widehat{D}_{n+1})\}$  and has defining relations

$$(4.1) \quad K_0 = 1, \quad K_\mu K_\lambda = K_{\mu+\lambda},$$

$$(4.2) \quad K_\mu E_i = q^{\langle \mu, \alpha_i \rangle} E_i K_\mu, \quad K_\mu F_i = q^{-\langle \mu, \alpha_i \rangle} F_i K_\mu,$$

$$(4.3) \quad E_i E_j = E_j E_i, \quad F_i F_j = F_j F_i, \quad (\langle \alpha_i, \alpha_j \rangle = 0 \text{ or } 2),$$

$$(4.4) \quad E_i [E_i, E_j] = q [E_i, E_j] E_i, \quad F_i [F_i, F_j] = q [F_i, F_j] F_i, \quad (\langle \alpha_i, \alpha_j \rangle = -1),$$

$$(4.5) \quad E_i F_j = F_j E_i + \frac{\delta_{ij}}{q} (K_{\alpha_i} - K_{-\alpha_i}),$$

for every  $i, j \in \{0, \dots, n+1\}$  and  $\mu, \lambda \in Q(\widehat{D}_{n+1})$  (c.f. Eqns. 3.1-3.5).

Let  $\widehat{w}_n \in W(\widehat{D}_{n+1})$  be the Weyl group element given by

$$(4.6) \quad \widehat{w}_n : v + r \mapsto v + r + 2a_{n+1}$$

for every  $v = \sum_{i=1}^{n+1} a_i e_i \in Q(D_{n+1})$  and  $r \in \mathbb{Z}$ . We have the reduced expression

$$(4.7) \quad \widehat{w}_n := (s_{n+1} \cdots s_1)(s_3 \cdots s_{n+1})s_0(s_n \cdots s_3)(s_1 \cdots s_n)s_0 \in W(\widehat{D}_{n+1}).$$

We let  $\widehat{B}_{\mathfrak{so}_{2n+2}} = \langle T_0, \dots, T_{n+1} \rangle$  denote the corresponding braid group of  $\widehat{\mathfrak{so}}_{2n+2}$  and label the corresponding ordered root vectors for  $\mathcal{U}_q^{\widehat{w}_n}$  by

$$(4.8) \quad X_n, \dots, X_1, Y_1, \dots, Y_n, \overline{X}_n, \dots, \overline{X}_1, \overline{Y}_1, \dots, \overline{Y}_n.$$

One can readily verify the following lemmas.

**Lemma 4.1.** *We have the following recursion formulas in the algebra  $\mathcal{U}_q^{\widehat{w}_n}$ :*

$$(4.9) \quad X_n = E_{n+1}, \quad X_i = [X_{i+1}, E_{i+1}], \quad (i \neq n),$$

$$(4.10) \quad Y_1 = [X_2, E_1], \quad Y_i = [Y_{i-1}, E_i], \quad (i \neq 1),$$

$$(4.11) \quad \overline{X}_n = [Y_{n-1}, T_{n+1} T_n E_0], \quad \overline{X}_i = [\overline{X}_{i+1}, E_{i+1}], \quad (i \neq n),$$

$$(4.12) \quad \overline{Y}_1 = [\overline{X}_2, E_1], \quad \overline{Y}_i = [\overline{Y}_{i-1}, E_i], \quad (i \neq 1),$$

$$(4.13) \quad Y_2 = [X_1, E_1], \quad \overline{Y}_2 = [\overline{X}_1, E_1].$$

**Lemma 4.2.** *For all  $i, j \in \{1, \dots, n\}$ , we have the following:*

$$(4.14) \quad T_i.X_j = \begin{cases} [E_i, X_j], & (i = j \text{ or } (i, j) = (1, 2)), \\ X_{j+1}, & (i = j + 1), \\ X_j, & \text{otherwise}, \end{cases}$$

$$(4.15) \quad T_i.Y_j = \begin{cases} Y_{j-1}, & (i = j \text{ and } i \neq 1), \\ X_{3-j}, & (i = 1, j \in \{1, 2\}), \\ [E_{j+1}, Y_j], & (i = j + 1), \\ Y_j, & \text{otherwise}, \end{cases}$$

$$(4.16) \quad T_i.\overline{X}_j = \begin{cases} [E_i, \overline{X}_j], & (i = j \text{ or } (i, j) = (1, 2)), \\ \overline{X}_{j+1}, & (i = j + 1), \\ \overline{X}_j, & \text{otherwise}, \end{cases}$$

$$(4.17) \quad T_i.\overline{Y}_j = \begin{cases} \overline{Y}_{j-1}, & (i = j \text{ and } i \neq 1), \\ \overline{X}_{3-j}, & (i = 1, j \in \{1, 2\}), \\ [E_{j+1}, \overline{Y}_j], & (i = j + 1), \\ \overline{Y}_j, & \text{otherwise}. \end{cases}$$

With the help of Lemmas 4.1 and 4.2, we prove the following.

**Proposition 4.3.** *The defining relations for the algebra  $\mathcal{U}_q^{\widehat{w}_n}$  are*

$$(4.18) \quad X_i X_j = q^{-1} X_j X_i, \quad Y_j Y_i = q^{-1} Y_i Y_j, \quad (i < j),$$

$$(4.19) \quad \overline{X}_i \overline{X}_j = q^{-1} \overline{X}_j \overline{X}_i, \quad \overline{Y}_j \overline{Y}_i = q^{-1} \overline{Y}_i \overline{Y}_j, \quad (i < j),$$

$$(4.20) \quad Y_j X_i = q^{\delta_{ij}-1} X_i Y_j - \delta_{ij} \hat{q} \sum_{r=1}^{i-1} (-q)^{i-r-1} X_r Y_r,$$

$$(4.21) \quad \overline{Y}_j \overline{X}_i = q^{\delta_{ij}-1} \overline{X}_i \overline{Y}_j - \delta_{ij} \hat{q} \sum_{r=1}^{i-1} (-q)^{i-r-1} \overline{X}_r \overline{Y}_r,$$

$$(4.22) \quad \overline{X}_i X_i = q^{-2} X_i \overline{X}_i, \quad \overline{Y}_i Y_i = q^{-2} Y_i \overline{Y}_i,$$

$$(4.23) \quad \overline{X}_j X_i = q^{-1} X_i \overline{X}_j, \quad \overline{Y}_i Y_j = q^{-1} Y_j \overline{Y}_i, \quad (i < j),$$

$$(4.24) \quad \overline{X}_i X_j = q^{-1} X_j \overline{X}_i - q^{-1} \hat{q} X_i \overline{X}_j, \quad \overline{Y}_j Y_i = q^{-1} Y_i \overline{Y}_j - q^{-1} \hat{q} Y_j \overline{Y}_i, \quad (i < j),$$

$$(4.25) \quad \overline{X}_i Y_j = q^{-1+\delta_{ij}} Y_j \overline{X}_i + \hat{q} q^{-1} \delta_{ij} \sum_{m=i+1}^n (-q)^{m-i} Y_m \overline{X}_m,$$

$$(4.26) \quad \overline{Y}_i X_j = q^{-1+\delta_{ij}} X_j \overline{Y}_i - \hat{q} q^{-1} Y_i \overline{X}_j \\ + \hat{q} q^{-1} \delta_{ij} \left[ \sum_{m=1}^n (-q)^{i+m-2} Y_m \overline{X}_m + \sum_{m=1}^{i-1} (-q)^{i-m} X_m \overline{Y}_m \right],$$

for  $i, j \in \{1, \dots, n\}$ .

**PROOF.** The first  $2n$  letters in the reduced expression for  $\widehat{w}_n$  coincide with  $w_n$ , as do the last  $2n$  letters. This gives us the relations 4.18-4.21. Using Lemmas 4.1 and 4.2, one can prove inductively that the remaining relations hold. To illustrate how to obtain the identities in Eqn. 4.22 for example, one can first verify the base cases,  $\overline{X}_1 X_1 = q^{-2} X_1 \overline{X}_1$  and  $\overline{Y}_n Y_n = q^{-2} Y_n \overline{Y}_n$ , and then apply appropriate

braid group automorphisms (refer to Lemma 4.2) to both sides of the equations. Since  $\mathcal{U}_q^{\widehat{w}_n}$  has a PBW basis of ordered monomials, Eqns. 4.18-4.26 are the defining relations.  $\square$

By comparing Eqns. 3.28-3.37 with Eqns. 4.18-4.26, we observe the following theorem.

**Theorem 4.4.** *As  $k$ -algebras,  $\mathcal{U}_q^{\widehat{w}_n} \cong (\mathcal{U}_q^{w_n})^\#$  via the isomorphism*

$$\begin{aligned} X_i &\mapsto (x_i \# 1), & Y_i &\mapsto (y_i \# 1), \\ \overline{X}_i &\mapsto (1 \# x_i), & \overline{Y}_i &\mapsto (1 \# y_i), \end{aligned} \quad \text{for } i = 1, \dots, n.$$

## 5. The FRT-Construction and the algebra $\mathbb{X}_{n,q}$

We will briefly review the Faddeev-Reshetikhin-Takhtajan (FRT) construction of [7] (see [5, Section 7.2] for more details). We let  $V$  be a  $k$ -module with basis  $\{v_1, \dots, v_N\}$ . For a linear map  $R \in \text{End}_k(V \otimes V)$ , we write

$$(5.1) \quad R(v_i \otimes v_j) = \sum_{s,t} R_{ij}^{st} v_s \otimes v_t \text{ for all } 1 \leq i, j < N,$$

with all  $R_{ij}^{st} \in k$ . The *FRT algebra*  $\mathcal{A}(R)$  associated to  $R$  is the  $k$ -algebra presented by generators  $X_{ij}$  for  $1 \leq i, j \leq N$  and has the defining relations

$$(5.2) \quad \sum_{s,t} R_{st}^{ji} X_{sl} X_{tm} = \sum_{s,t} R_{lm}^{ts} X_{is} X_{jt}$$

for every  $i, j, l, m \in \{1, \dots, N\}$ . Up to algebra isomorphism,  $\mathcal{A}(R)$  is independent of the chosen basis of  $V$ .

Let us specialize now to the case when  $N = 2n$ . Following [12, Section 8.4.2], for each  $i, j \in \{1, \dots, 2n\}$ , let  $E_{ij}$  denote the linear map on  $V$  defined by  $E_{ij}.v_\ell = \delta_{j\ell} v_i$ . Let  $i' := 2n + 1 - i$ , and let

$$(5.3) \quad \begin{aligned} R_{D_n} = q \sum_{i:i \neq i'} (E_{ii} \otimes E_{ii}) + \sum_{i,j:i \neq j,j'} (E_{ii} \otimes E_{jj}) + q^{-1} \sum_{i:i \neq i'} (E_{i'i'} \otimes E_{ii}) \\ + \hat{q} \left( \sum_{i,j:i > j} (E_{ij} \otimes E_{ji}) - \sum_{i,j:i > j} q^{\rho_i - \rho_j} (E_{ij} \otimes E_{i'j'}) \right), \end{aligned}$$

where  $(\rho_1, \rho_2, \dots, \rho_{2n})$  is the  $2n$ -tuple  $(n-1, n-2, \dots, 1, 0, 0, -1, \dots, -n+1)$ .

We define an algebra  $\mathbb{X}_{n,q}$  presented by generators  $X_{ij}$  with  $i \in \{1, 2\}$ ,  $j \in \{1, \dots, 2n\}$ , and having the defining relations

$$(5.4) \quad X_{rt}X_{rs} = q^{-1}X_{rs}X_{rt} \quad (r \in \{1, 2\}, s < t, t \neq s'),$$

$$(5.5) \quad X_{rs'}X_{rs} = X_{rs}X_{rs'} + \hat{q} \sum_{l=s+1}^n q^{l-s-1} X_{rl}X_{rl'} \quad (r \in \{1, 2\}, s < s'),$$

$$(5.6) \quad X_{2s}X_{1s} = q^{-1}X_{1s}X_{2s},$$

$$(5.7) \quad X_{2s}X_{1t} = X_{1t}X_{2s} \quad (s < t, t \neq s'),$$

$$(5.8) \quad X_{2t}X_{1s} = X_{1s}X_{2t} - \hat{q}X_{1t}X_{2s} \quad (s < t, t \neq s'),$$

$$(5.9) \quad X_{2s}X_{1s'} = qX_{1s'}X_{2s} + \hat{q} \sum_{l=1}^{s-1} q^{s-l} X_{1l'}X_{2l} \quad (s < s'),$$

$$(5.10) \quad X_{2s'}X_{1s} = qX_{1s}X_{2s'} + \hat{q} \sum_{l=s+1}^n q^{l-s} X_{1l}X_{2l'},$$

$$+ \hat{q}q^{-1} \sum_{l=1}^n q^{l'-s} X_{1l'}X_{2l} - \hat{q}X_{1s'}X_{2s} \quad (s < s').$$

We label the canonical generators of  $\mathcal{A}(R_{D_n})$  by  $Y_{ij}$  for  $i, j = 1, \dots, 2n$ , and let  $T_{2,n}$  be the subalgebra of  $\mathcal{A}(R_{D_n})$  generated by  $\{Y_{ij} : 1 \leq i \leq 2, 1 \leq j \leq 2n\}$ .

**Proposition 5.1.** *There is a surjective algebra homomorphism  $\mathbb{X}_{n,q} \rightarrow T_{2,n}$  with kernel  $\langle \Omega_1, \Omega_2, \Upsilon \rangle$ , where*

$$(5.11) \quad \Omega_1 := \sum_{r=1}^n q^{\rho_{r'}} X_{1,r}X_{1,r'}, \quad \Omega_2 := \sum_{r=1}^n q^{\rho_{r'}} X_{2,r}X_{2,r'}, \quad \Upsilon := \sum_{r=1}^{2n} q^{\rho_r} X_{1,r'}X_{2,r}.$$

PROOF. Using the FRT construction (see Equation 5.2 and 5.3), one can readily compute the defining relations for the algebra  $\mathcal{A}(R_{D_n})$  and see that they line up appropriately with Equation 5.4-5.10 together with  $\Omega_1 = \Omega_2 = \Upsilon = 0$ .  $\square$

Notice that the definition of  $\mathbb{X}_{n,q}$  makes sense when  $n = 2$ , and Proposition 5.1 holds in this case as well. However, the rest of the results of this paper require  $n \geq 3$ .

Following [1], we recall the details on twisting algebras by cocycles. Let  $M$  be an additive abelian group and  $c : M \times M \rightarrow k^\times$  a 2-cocycle of  $M$ . If  $\Lambda$  is a  $k$ -algebra graded by  $M$ , we can twist  $\Lambda$  by  $c$  to obtain a new  $M$ -graded  $k$ -algebra  $\Lambda'$  that is canonically isomorphic to  $\Lambda$  as a  $k$ -module via  $x \leftrightarrow x'$ . Multiplication of homogeneous elements in  $\Lambda'$  is given by

$$x'y' = c(\deg(x), \deg(y))(xy)'.$$

For our purposes, we will let  $\beta : Q(\widehat{D}_{n+1}) \times Q(\widehat{D}_{n+1}) \rightarrow k^\times$  be the bicharacter (hence, also a 2-cocycle) defined by

$$(5.12) \quad \beta(\alpha_i, \alpha_j) = \begin{cases} q & (i, j) = (0, n+1), \\ 1 & (i, j) \neq (0, n+1), \end{cases}$$

and have the following:

**Theorem 5.2.** *The  $\beta$ -twisted algebra  $(\mathcal{U}_q^{\widehat{w}_n})'$  is isomorphic to  $\mathbb{X}_{n,q}$ .*

PROOF. We label the corresponding generators of  $(\mathcal{U}_q^{\widehat{w}_n})'$  by

$$(5.13) \quad X'_n, \dots, X'_1, Y'_1, \dots, Y'_n, \overline{X}'_n, \dots, \overline{X}'_1, \overline{Y}'_1, \dots, \overline{Y}'_n.$$

By comparing Eqns. 4.18-4.26 and 5.4-5.10, we observe that the algebra map  $(\mathcal{U}_q^{\widehat{w}_n})' \rightarrow \mathbb{X}_{n,q}$  defined by

$$(5.14) \quad X'_i \mapsto (-1)^{n+1-i} X_{1,n+1-i}, \quad \overline{X}'_i \mapsto (-1)^{n+1-i} X_{2,n+1-i},$$

$$(5.15) \quad Y'_i \mapsto X_{1,n+i}, \quad \overline{Y}'_i \mapsto X_{2,n+i},$$

for every  $i \in \{1, \dots, n\}$ , is an isomorphism.  $\square$

From this, we deduce the following:

**Theorem 5.3.** *The algebra  $\mathbb{X}_{n,q}$  is an iterated Ore extension over  $k$ ,*

$$\mathbb{X}_{n,q} = k[X_{11}][X_{12}; \tau_{12}, \delta_{12}] \cdots [X_{1n}; \tau_{1n}, \delta_{1n}][X_{21}; \tau_{21}, \delta_{21}] \cdots [X_{2n}; \tau_{2n}, \delta_{2n}].$$

PROOF. It suffices to check that ordered monomials are linearly independent. From Theorem 5.2, we have a canonical vector space isomorphism  $\mathcal{U}_q^{\widehat{w}_n} \rightarrow \mathbb{X}_{n,q}$  that preserves the ordered generating sets. Since  $\mathcal{U}_q^{\widehat{w}_n}$  has a basis of ordered monomials,  $\mathbb{X}_{n,q}$  does as well.  $\square$

## 6. A type $A_m$ analogue

**6.1. The algebras  $\mathcal{U}_q(\mathfrak{sl}_{m+1})$ ,  $\mathcal{U}_q^{c_m}$ , and  $\mathcal{O}_q(k^m)$ .** Fix an integer  $m > 1$ . Let  $Q(A_m)$  denote the abelian subgroup of  $\mathbb{R}_{m+1}$  consisting of integral  $(m+1)$ -tuples  $(a_1, \dots, a_{m+1})$  with the sum  $\sum a_i$  equalling 0. As a group,  $Q(A_m)$  is generated by  $\alpha_i := e_i - e_{i+1}$  for  $i \in \{1, \dots, m\}$ . Let  $W(A_m)$  and  $B_{\mathfrak{sl}_{m+1}}$  denote the corresponding Weyl group and braid group, respectively. Let  $\mathcal{U}_q(\mathfrak{sl}_{m+1})$  denote the corresponding quantum enveloping algebra, and let  $\mathcal{U}_A^{\geq 0}$  be the positive Borel subalgebra of  $\mathcal{U}_q(\mathfrak{sl}_{m+1})$ . We consider the Coxeter element

$$(6.1) \quad c_m = s_1 \cdots s_m \in W(A_m)$$

and the associated De Concini-Kac-Procesi algebra  $\mathcal{U}_q^{c_m}$ . We label the root vectors in  $\mathcal{U}_q^{c_m}$  by

$$(6.2) \quad z_1 := X_{e_1 - e_2}, \quad z_2 := X_{e_1 - e_3}, \dots, z_m := X_{e_1 - e_{m+1}}$$

and have the following

**Proposition 6.1.** *The root vectors  $z_1, \dots, z_m$  satisfy the relations*

$$(6.3) \quad z_i z_j = q z_j z_i$$

for all  $i, j \in \{1, \dots, m\}$  with  $i < j$ .

Since  $\mathcal{U}_q^{c_m}$  has a PBW basis of ordered monomials, the relations of Eqn. 6.3 are the defining relations for  $\mathcal{U}_q^{c_m}$ . In particular, we have the following well-known result (c.f. for example [15]):

**Corollary 6.2.** *The algebra  $\mathcal{U}_q^{c_m}$  is isomorphic to the algebra of quantum affine space  $\mathcal{O}_q(k^m)$ .*

Denote by  $\pi_A : \mathcal{U}_A^{\geq 0} \rightarrow \mathcal{U}_A^{\geq 0}$  the unique algebra map such that

$$(6.4) \quad \pi(E_1) = 0,$$

$$(6.5) \quad \pi(E_i) = E_i \quad (1 < i \leq m),$$

$$(6.6) \quad \pi(K_\mu) = K_\mu \quad (\mu \in Q(A_m)).$$

Let  $\lambda_A : \mathcal{U}_A^{\geq 0} \otimes \mathcal{U}_q^{c_m} \rightarrow \mathcal{U}_A^{\geq 0}$  be defined by the following sequence of linear maps:

$$(6.7) \quad \lambda_A : \mathcal{U}_A^{\geq 0} \otimes \mathcal{U}_q^{c_m} \xrightarrow{\text{incl.}} \left(\mathcal{U}_A^{\geq 0}\right)^{\otimes 2} \xrightarrow{\pi_A \otimes \text{id}} \left(\mathcal{U}_A^{\geq 0}\right)^{\otimes 2} \xrightarrow{\text{adjoint}} \mathcal{U}_A^{\geq 0}.$$

The identities in Equation 3.16 imply the following

**Corollary 6.3.** *The linear map  $\lambda_A$  satisfies  $\text{Im}(\lambda_A) \subseteq \mathcal{U}_q^{c_m}$ . In particular,  $\lambda_A$  endows the algebra  $\mathcal{U}_q^{c_m}$  with the structure of a left  $\mathcal{U}_A^{\geq 0}$ -module algebra.*

As before (see 3.18), we use the action map  $\lambda_A$  to construct the smash product  $\mathcal{U}_q^{c_m} \# \mathcal{U}_A^{\geq 0}$  and let  $(\mathcal{U}_q^{c_m})^\#$  denote the subalgebra

$$(6.8) \quad (\mathcal{U}_q^{c_m})^\# := \langle 1 \# u, u \# 1 \mid u \in \mathcal{U}_q^{c_m} \rangle \subseteq \mathcal{U}_q^{c_m} \# \mathcal{U}_A^{\geq 0}.$$

**6.2. The quantum affine algebra  $\mathcal{U}_q^{\widehat{c}_m}$ .** Let  $Q(\widehat{A}_m) = Q(A_m) \oplus \mathbb{Z}$  denote the root lattice of type  $\widehat{A}_m$ . As an abelian group,  $Q(\widehat{A}_m)$  is generated additively by the positive simple roots  $\alpha_0 := e_m - e_1 + 1$ , and  $\alpha_i := e_i - e_{i+1}$  for  $i \in \{1, \dots, m\}$ . We extend the inner product  $\langle \cdot, \cdot \rangle$  on  $Q(A_m)$  to an inner product on  $Q(\widehat{A}_m)$  by setting  $1 \in Q(\widehat{A}_m)$  to be isotropic. We let  $s_i$  denote the corresponding simple reflection  $s_i : Q(\widehat{A}_m) \rightarrow Q(\widehat{A}_m)$ , for  $0 \leq i \leq m$ , and let  $W(\widehat{A}_m) = \langle s_0, \dots, s_m \rangle$  denote the corresponding affine Weyl group. We let  $\mathcal{U}_q(\widehat{\mathfrak{sl}}_{m+1})$  denote the corresponding quantized enveloping algebra.

We set

$$(6.9) \quad \widehat{c}_m := (s_1 \cdots s_m)(s_0 s_1 \cdots s_{m-1}) \in W(\widehat{A}_m)$$

and note the following analogue of Theorem 4.4.

**Theorem 6.4.** *As  $k$ -algebras,  $\mathcal{U}_q^{\widehat{c}_m} \cong (\mathcal{U}_q^{c_m})^\#$ .*

PROOF. Compute. One can use an analogous isomorphism of Thm. 4.4.  $\square$

Now let  $V$  be a  $k$ -module with basis  $\{v_1, \dots, v_m\}$ , and for all  $i, j, \ell \in \{1, \dots, m\}$ , define linear maps  $e_{ij}$  by the rule  $e_{ij}.v_\ell = \delta_{j\ell}v_i$ .

Set

$$(6.10) \quad R_{A_{m-1}} = q \sum_{i=1}^m (e_{ii} \otimes e_{ii}) + \sum_{i \neq j} (e_{ii} \otimes e_{jj}) + \hat{q} \sum_{i > j} (e_{ij} \otimes e_{ji}).$$

This is the standard  $R$ -matrix of type  $A_{m-1}$  (see [12, Section 8.4.2]).

The algebra of  $m \times m$  quantum matrices, denoted  $\mathcal{O}_q(M_m(k))$ , is the algebra  $\mathcal{A}(R_{A_{m-1}})$  and was defined in [7]. More generally, one considers  $\ell \times p$  quantum matrices, denoted  $\mathcal{O}_q(M_{\ell,p}(k))$ , by looking at appropriate subalgebras of square quantum matrices.

We let  $\gamma : Q(\widehat{A}_m) \times Q(\widehat{A}_m) \rightarrow k^\times$  be the bicharacter defined by

$$(6.11) \quad \gamma(\alpha_i, \alpha_j) = \begin{cases} q, & (i, j) = (0, 1), \\ 1, & (i, j) \neq (0, 1). \end{cases}$$

and have the following analogue of Thm. 5.2.

**Theorem 6.5.** *Twisting  $\mathcal{U}_q^{\widehat{C}_m}$  by  $\gamma$  yields an algebra isomorphic to  $\mathcal{O}_q(M_{2,m})$ .*

PROOF. Compute (c.f. Thm. 5.2). □

Theorem 6.5, together with Proposition 5.1, allows us to view  $\mathbb{X}_{n,q}$  as an orthogonal analogue of  $2 \times n$  quantum matrices. The key distinction is that  $\mathcal{O}_q(M_{2,n}(k))$  is a subalgebra of  $\mathcal{A}(R_{A_{n-1}})$ , whereas  $\mathbb{X}_{n,q}$  is a parent of the analogous subalgebra  $T_{2,n} \subseteq \mathcal{A}(R_{D_n})$ .

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